Part I: Covers Sequence through Series Comparison Tests

1. Give an example of each of the following:

(a) A geometric sequence:

(b) An alternating sequence:

(c) A sequence that is bounded, but not convergent:

(d) A sequence that is monotonic, but not convergent:

(e) A sequence that is not bounded and nor monotonic:
2. For each of the sequences below, determine if they are convergent. If they are, find their limit. Remember a few things:

i) If the sequence looks like something we could have taken the limit of in Calculus 1, then you can find the limit the same way you would have in Calculus 1.

ii) Factorials grow faster than polynomials and exponential functions, but not as fast as things that look like $n^n$.

iii) If the sequences is recursive, you usually determine convergence by trying to show the sequence is bounded and monotonic. If it is, you can then find its limit $L$ from the recursion formula, by replacing each instance of the sequence with the limit $L$ and then solving for $L$.

(a) $a_n = \frac{4n^3 + 2n + 4}{5n^3 - 100n + 2}$

(b) $b_n = \frac{2^n}{n^3}$

(c) $d_n = \ln(3n + 1) - \ln(n)$

3. What is $\sqrt{3 + \sqrt{3 + \sqrt{3 + \ldots}}}$? (it is bounded above)
4. For each of the series below, compute the first 5 partial sums $S_1$ through $S_5$. Do you think these series converge or diverge?

(a) $\sum_{n=1}^{\infty} \frac{n - 1}{2n + 1}$

(b) $\sum_{n=1}^{\infty} \frac{1}{1,000,000}$

(c) $\sum_{n=1}^{\infty} \frac{1000}{10^n}$

5. Which of the following would change whether or not a given series converges?

- Starting the series at $n = 5$ instead of $n = 1$.
- Deleting a million terms from the series.
- Adding a million terms to the series.
6. Does the sum of two convergent series always converge? Does the sum of two divergent series always diverge? What happens if you add a convergent series and a divergent series together?

7. For each of the following series, determine if they converge or diverge. If they converge, use the geometric series formula and/or the telescoping sum techniques to compute the sum exactly.

(a) \( \sum_{n=2}^{\infty} \frac{1 + 2^{n+1}}{3^n} \)

(b) \( \sum_{n=10}^{\infty} \frac{1 + 4^{n-1}}{3^n} \)

(c) \( \sum_{n=10}^{\infty} \frac{1}{\arctan n} \)
(d) \( \sum_{n=1}^{\infty} \frac{3}{n^2 + 5n + 4} \)

(e) \( \sum_{n=1}^{\infty} \frac{7}{5^n} + \frac{3}{n^2 + n} \)
8. One application of summing geometric series is to convert repeating decimals into their fractional counterparts. For instance, the decimal 0.7 is really just the infinite series
\[ \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \ldots \]
Which we can rewrite as
\[ \sum_{n=1}^{\infty} \frac{7}{10^n} \]
Summing this series, we obtain that the above repeating decimal is just \( \frac{7}{9} \).
Use this to find the fractional representations of the repeating decimals below:

(a) 0.\overline{9}

(b) 0.\overline{16}
9. The Integral Test: The integral test says that if \( f(n) \) is a sequence that satisfies a couple of properties, then \( \sum_{n=0}^{\infty} f(n) \) converges. What are these properties?

If \( g(x) \) is continuous and decreasing:

10. If \( 0 \leq g(x) \leq f(x) \) and \( \int_k^\infty g(x) \, dx \) converges, then \( \sum_{n=k}^{\infty} f(n) \) converges. (T/F)

11. If \( 0 \leq f(x) \leq g(x) \) and \( \int_k^\infty g(x) \, dx \) converges, then \( \sum_{n=k}^{\infty} f(n) \) converges. (T/F)

12. Since \( \int_1^\infty \frac{1}{x^2} \, dx = 1 \), does this mean that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 \)? Why or why not? If not, can we find the sum of \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) using the techniques we learned in this class?
13. Use the Integral Test to determine whether each of these series converge or diverge:

(a) \[ \sum_{n=1}^{\infty} \frac{e}{1 + n^2} \]

(b) Use the result from above, determine if \[ \sum_{n=1}^{\infty} \frac{e^{1/n}}{1 + n^2} \] converges.
14. The p-test: Sums of the form \( \sum_{n=1}^{\infty} \frac{1}{n^p} \), or \( \sum_{n=1}^{\infty} \frac{1}{n(\ln(n))^p} \) commonly appear in applications. These can be handled by the Integral Test; however, whether or not they converge due the Integral Test depends on the value of \( p \). Rather than do the Integral Test every single time, we just remember the values of \( p \) for which these kinds of series converge. Fill in the blanks below:

- If \( p \leq 1 \), then \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) ________________
- If \( p > 1 \), then \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) ________________
- If ________, then \( \sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p} \) diverges.
- If ________, then \( \sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p} \) converges.

15. Use the \( p \)-test to quickly determine if the following integrals converge or diverge:

(a) \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \)
(b) \( \sum_{n=8}^{\infty} \frac{1}{n^{1/2} \ln n} \)
(c) \( \sum_{n=8}^{\infty} \frac{1}{n(\ln n)^{3/2}} \)
(d) \( \sum_{n=8}^{\infty} \frac{1}{n^2(\ln n)} \)
(e) \( \sum_{n=8}^{\infty} \frac{1}{n(\ln n)} \)
16. **Working With Inequalities: A Primer**

When using the Direct Comparison Test to show that a series \( \sum a_n \), where \( a_n > 0 \), converges or diverges, you are looking to find another sequence \( b_n \), where \( b_n > 0 \), such that:

- \( a_n \leq b_n \) and \( \sum b_n \) converges. This shows that \( \sum a_n \) converges.

or that:

- \( b_n \leq a_n \) and \( \sum b_n \) diverges. This shows that \( \sum a_n \) diverges.

But how do you show that \( a_n \leq b_n \) or that \( b_n \leq a_n \)? Usually, you will have a guess for what \( b_n \) should be. Then, starting with \( a_n \), you will manipulate the formula for \( a_n \) in ways that will make it either larger or smaller until you reach your target. The following page will explain some ways to do this.

**How to make something bigger**

- Add something positive to it (Ex: \( a_n \leq a_n + 1 \))
- Remove something negative from it. (Ex: \( a_n - 1 \leq a_n \))
- Replace a term with something bigger. (Ex: \( n^2 + n \leq n^2 + n^2 = 2n^2 \))
- If you have a fraction, make the numerator larger.
- If you have a fraction, make the denominator smaller.
- If you have some other increasing function, such as a square root, you can make the stuff inside of it larger. (Ex: \( \sqrt{n} < \sqrt{n + 1} \)).

**How to make something smaller**

- Add something negative to it (Ex: \( a_n \geq a_n - 1 \))
- Remove something positive from it. (Ex: \( a_n + 1 \geq a_n \))
- Replace a term with something smaller. (Ex: \( n^2 + \sin(n) \geq n^2 - 1 \))
- If you have a fraction, make the numerator smaller.
- If you have a fraction, make the denominator larger.
- If you have some other increasing function, such as a square root, you can make the stuff inside of it smaller. (Ex: \( \sqrt{n + 1} > \sqrt{n} \)).
Strategy for Direct Comparison

- Identify the terms that grow the fastest in both the numerator and denominator. Their ratio will usually form the sequence $b_n$. Determine if $\sum b_n$ converges or diverges.
- If $b_n$ converges, then use the “make something bigger” strategies above to remove terms or replace terms of $a_n$ until you get $b_n$.
- If $b_n$ diverges, then use the “make something smaller” strategies above to remove terms or replace terms of $a_n$ until you get $b_n$.

Example: Determine the convergence of $\sum_{n=1}^{\infty} \frac{\sqrt{n^2 + 1}}{3n^3 - n^2 + n + 1}$.

The largest term in the numerator is $\sqrt{n^2}$, and the largest in the denominator is $n^3$. So this should behave like $\frac{\sqrt{n^2}}{n^3} = \frac{n}{n^3} = \frac{1}{n^2}$, and $\sum \frac{1}{n^2}$ converges. So we want to make $a_n = \frac{\sqrt{n^2 + 1}}{3n^3 - n^2 + n + 1}$ larger until we get to $b_n = \frac{1}{n^2}$.

Using our rules above, we have that

\[
a_n = \frac{\sqrt{n^2 + 1}}{3n^3 - n^2 + n + 1} \leq \frac{\sqrt{n^2 + n^2}}{3n^3 - n^2 + n + 1} \leq \frac{\sqrt{2n^2}}{3n^3 - n^2} \leq \frac{\sqrt{2n^2}}{3n^3 - n^3} = \frac{\sqrt{2n}}{2n^3} = \frac{\sqrt{2}}{n^2} = \sqrt{2}b_n
\]

The $\sqrt{2}$ in front of the $b_n$ doesn’t matter, as $\sum \sqrt{2}b_n$ converges if $\sum b_n$ does. Thus, since $\sum \sqrt{2}b_n$ converges, so does $\sum a_n$.

The point of the lengthy discussion above is to show you that in order to use the direct comparison test, it is very tedious to find the correct ‘size’ for $b_n$. I hope it’s enough to convenience you that, unless you NEED to use the direct comparison test when it’s easy to find an upper and lower bounds for the $a_n$, it’s much easier to use the limit comparison where you do not have to deal with the ‘size’.
17. Determine the convergence of each of the following series using Direct Comparison:

(a) \[ \sum_{n=1}^{\infty} \frac{3^n + n}{2^n - n^3} \]

(b) \[ \sum_{n=2}^{\infty} \frac{1}{n^2 \ln n} \]

(c) \[ \sum_{n=1}^{\infty} \frac{10 - \cos n}{2^n} \]

(d) \[ \sum_{n=1}^{\infty} \frac{(-1)^n + 4}{n \ln n} \]
**Limit Comparison:** Sometimes, though, you will know what you want to compare your series to, but you cannot get Direct Comparison to work. (Or work easily, in any case). In this case, you should try Limit Comparison instead of Direct Comparison. The steps are pretty similar:

- Like you would for Direct Comparison, identify the terms that grow the fastest in both the numerator and denominator. Their ratio will usually form the sequence $b_n$. (We may phrase this as figuring out what the series “behaves like”).

- Find $\lim_{n \to \infty} \frac{a_n}{b_n}$. If this limit exists, is finite, and is larger than 0, then the two series $\sum a_n$ and $\sum b_n$ both converge or both diverge. Essentially, this limit calculation shows that the two sequences decay at the same rate, and so their series will grow at the same rate.

- Limit Comparison is often better than Direct Comparison for series involving complicated rational functions, as well as series where you don’t have a good guess for something to compare directly with.

18. If $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges. Explain why this is the case.

19. Similarly, if $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ and $\sum a_n$ converges, then $\sum b_n$ converges. Explain why this is the case.
20. For each of the positive-term series $\sum a_n$ below, we can determine convergence by comparing to another series $\sum b_n$. Choose the sequence $b_n$ to compare to, say whether $\sum b_n$ converges or diverges, and indicate whether you would use the Direct Comparison Test or the Limit Comparison Test to carry out the justification. Also give the inequality you will use for Direct Comparison Test, or the value of the limit, for the Limit Comparison Test.

(a) $\sum_{n=1}^{\infty} \frac{n^3 + 2n - 1}{n^5 - 2n + 3}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n} \cdot \sin \left( \frac{1}{\sqrt[3]{n}} \right)$

(c) $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^4 + 2n + 1}}$

(d) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

(e) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

(f) $\sum_{n=1}^{\infty} \sin \left( \frac{1}{\sqrt{n}} \right)$

(g) $\sum_{n=2}^{\infty} \frac{\arctan n}{\sqrt{n^3 - 1}}$

(h) $\sum_{n=1}^{\infty} \frac{\sin^2 n}{4n^5 + n}$
(a) \[ \sum_{n=1}^{\infty} \frac{5 + 3 \sin n}{n^3} \]

(b) \[ \sum_{n=2}^{\infty} \frac{5 + 3 \sin n}{n} \]

(c) \[ \sum_{n=1}^{\infty} \frac{5}{n^{2^n}} \]

(d) \[ \sum_{n=1}^{\infty} \tan^3 \left( \frac{1}{n} \right) \]
1. Determine whether the sequence converges or diverges. If it converges, find the limit

a. \( a_n = \frac{(2n-1)!}{(2n+1)!} \) 

b. \( a_n = \frac{\cos^2 n}{2^n} \)

c. \( a_n = \frac{(-1)^{n+1}\sqrt{n}}{\sqrt{n+2}} \)

d. \( a_n = \frac{n!}{2^n} \)

e. \( a_n = \arctan(\ln n) \)

f. \( a_n = \sqrt{n} \)

g. \( a_n = \ln(4n) - \ln(4n-1) \)

h. \( a_n = \frac{n}{(\ln n)^n} \)

i. \( a_n = (1 + \frac{3}{n})^{2n} \)

2. Find the limit \( L \) of the sequence, or say DIV. (the recursive sequences (a) and (d) are convergent).

a. \( a_1 = 1, \quad a_{n+1} = 3 - \frac{1}{a_n} \)

b. \( \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \ldots \right\} \)

c. \( \left\{ \frac{1}{1 \cdot 3}, \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4}, \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5}, \ldots \right\} \)

d. \( \left\{ \sqrt{5}, \sqrt{5\sqrt{5}}, \sqrt{5\sqrt{5\sqrt{5}}}, \ldots \right\} \)

3.

i. What happens to the series \( \sum_{n=1}^{\infty} a_n \) if \( \lim_{n \to \infty} a_n = 0? \)

ii. What happens to the series \( \sum_{n=1}^{\infty} a_n \) if \( \lim_{n \to \infty} a_n \neq 0? \)

iii. Suppose \( S_N = \sum_{n=1}^{N} a_n \) and that \( S_N = 5 - \frac{N}{2N!} \). What can be said about \( \lim_{n \to \infty} a_n, \sum_{N=1}^{\infty} S_N, \sum_{n=1}^{\infty} (a_n + 1)? \) Evaluate \( \sum_{i=3}^{6} a_i \).

iv. Suppose \( S_N = \arctan n \), then \( \lim_{n \to \infty} a_n = 0. \) TRUE/FALSE?
4. Suppose you know that $\sum a_n < \infty$, $\sum b_n$ diverges. Which statement below is TRUE?

i) If $a_n < c_n$, then $\sum c_n$ diverges.

ii) If $c_n < b_n$, then $\sum c_n$ converges.

iii) If $c_n > b_n$, then $\sum c_n$ diverges.

iv) If $c_n < a_n$, then $\sum c_n$ diverges.

5. $\sum \frac{42\sqrt{n}}{5n^2 - 9n + 1}$ can be shown convergent using DCT by comparing with $\sum \frac{42}{5\sqrt{n^3}}$. TRUE or FALSE?

6. Determine the value of $k$ for which the series $\sum_{n=1}^{\infty} \frac{n^3}{\sqrt{n^k+24n}}$ will converge. Write your answer in interval notation.

7. Does the series converge? If so, find the sum.

a. $\sum_{n=1}^{\infty} \frac{3^{n+3}}{5^n-1}$

b. $\sum_{n=0}^{\infty} \frac{2^n+5^n}{2^n 5^n}$

c. $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{5^n+2}$

d. $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

8. Which of the following statements are true?

I. $\sum_{n=5}^{\infty} \frac{1}{(\ln n)^{3/4}}$ converges by the Direct Comparison Test.

II. $\sum_{n=5}^{\infty} \frac{(-1)^n}{n^{(\ln n)^4}}$ converges by the Alternating Series Test.

III. $\sum_{n=5}^{\infty} \frac{1}{n^{(\ln n)^2}}$ converges by the Direct Comparison Test.
IV. $\sum_{n=5}^{\infty} \frac{2}{n^3 \ln n}$ converges by the Direct Comparison Test.

9. **Use Root Test**: Which of the following series converge? Find the correct limit of the test.

a. $\sum_{n=1}^{\infty} (10^{\frac{1}{n}} - 1)^n$

b. $\sum_{n=1}^{\infty} \left(\frac{n-7}{9n+10}\right)^n$

c. $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{(\ln n)^n}$

d. $\sum_{n=1}^{\infty} \frac{n^n}{2^n}$

e. $\sum_{n=1}^{\infty} \frac{n^2}{10^n+1}$

f. $\sum_{n=4}^{\infty} \left(1 - \frac{1}{n}\right)^{5n}$

g. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$

h. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^3}$

i. $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{n^2}$

9′. **Use Ratio Test**: Which of the following series converge? Find the correct limit of the test.

a. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n + 1)}{2 \cdot 5 \cdot 8 \cdots (3n + 2)}$

b. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$
10. **Alternating series test.** Determine if the series converges absolutely, conditionally, or diverges.

a. \( \sum_{n=1}^{\infty} (-1)^n \frac{1}{n \ln n} \)  

b. \( \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{(\ln n)^n} \)  

c. \( \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}(\ln n)} \)

11. According to the alternating series error estimation, what is the least upper estimates to the error by using the first 4 terms to approximate the sum? (\(|R_4| \leq \) ___)

a. \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \)  

b. \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!} \)

12. Determine whether the series is convergent or divergent using **Direct Comparison** or **Limit Comparison**.

a. \( \sum_{n=1}^{\infty} \frac{1+\cos n}{e^n} \)  

b. \( \sum_{n=1}^{\infty} \frac{n+1}{n^3+n} \)

c. \( \sum_{n=1}^{\infty} \sqrt{n} \tan \left( \frac{1}{n} \right) \)

d. \( \sum_{n=1}^{\infty} \frac{\tan \left( \frac{1}{n^2} \right)}{\ln n} \)

e. \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{n^2} \)

f. \( \sum_{n=1}^{\infty} \frac{n}{\ln^2 n} \)

g. \( \sum_{n=1}^{\infty} \frac{\ln^2 n}{n} \)

13. Does the series converge? What test(s) do you use?

a. \( \sum_{n=1}^{\infty} \tan \left( \frac{1}{n} \right) \)  

b. \( \sum_{n=1}^{\infty} \cos \left( \frac{1}{n} \right) \)

c. \( \sum_{n=2}^{\infty} \frac{1}{n \sqrt{n} \ln n} \)

d. \( \sum_{n=1}^{\infty} (-1)^n \cos \left( \frac{1}{n^2} \right) \)

e. \( \sum_{n=1}^{\infty} \sin^n \left( \frac{1}{\sqrt{n}} \right) \)

f. \( \sum_{n=1}^{\infty} \sin^3 \left( \frac{1}{n} \right) \)
14. Which of the following series are divergent by the **Test for Divergence**?

a. \( \sum_{n=1}^{\infty} n^{1/n} \)  
b. \( \sum_{n=1}^{\infty} \frac{n^2}{n^2 - 2n + 5} \)  
c. \( \sum_{n=1}^{\infty} \frac{n^2}{e^n} \)  
d. \( \sum_{n=1}^{\infty} n \tan \left( \frac{1}{n} \right) \)

15. Determine whether each series is absolutely convergent, conditionally convergent or divergent. Be clear in your argument and note what test(s) you use.

a. \( \sum_{n=5}^{\infty} \frac{(-1)^n \arctan n}{n^2} \)

b. \( \sum_{n=5}^{\infty} \frac{(-1)^{n+1}}{n^{2/3} \ln^{10} n} \)