MAC 2233, Survey of Calculus, Exam 3 Review

This exam covers lectures 21 – 29,

This review includes typical exam problems. It is not designed to be comprehensive, but to be representative of topics covered on the exam. Be sure to work Exam 3 in the binder from Target Copy, and Exam 3 from fall available from the Broward Teaching Center. There is also an exam with video solutions posted on the website teachingcenter.ufl.edu/vsi/. Try to work an exam in 90 minutes without notes or outside help.

1. Find the derivative of each of the following:
   (a) \( f(x) = 3^{2x-1} \)  (b) \( f(x) = \log_4(x^2 - x) \)

2. Find the slope of the tangent line to \( f(x) = \ln|2 - \ln x| \) at \( x = e \).

3. Find the slope of the tangent line to the curve given by \( \sqrt{3x - y} - e^{x+y} = 1 + \ln x \) at \( (1, -1) \).

4. Find the first three derivatives of \( f(x) = \ln(x^2 + 1) \).

5. Find each value of \( x \) at which \( f(x) = \ln\left(\frac{2x - 6}{\sqrt{x^2 + 3}}\right) \) has a horizontal tangent line.

6. Find \( f'(x) \) if \( f(x) = \ln\frac{e^{x-3} \sqrt{6 + 3x}}{(3x + 1)^2} \).

7. Use Logarithmic Differentiation to find the slope of the tangent line to \( f(x) = x\sqrt{x} \) at \( x = 4 \).

8. Find each value at which \( f(x) = \frac{x^4}{4} - \frac{x^3}{3} - x^2 \) has a relative maximum or minimum. Find the absolute extrema of \( f(x) \) on \([-2, 1]\).
9. Find all critical numbers and relative extrema of \( g(x) = \frac{1}{2\sqrt{x}} - \frac{1}{x^2} \).

10. Let \( f(x) = \frac{\sqrt{3}x - 2}{x} \).
   
   (a) Find \( f'(x) \) and write as a single fraction.
   
   (b) Find the equation of each horizontal and vertical tangent line of \( f(x) \).
   
   (c) Find each \( x \)-value at which \( f(x) \) has a critical number.
   
   (d) Find the relative extreme values of \( f(x) \).

11. The cost function for a product is \( C(x) = 1.25x^2 + 25x + 8000 \).
   
   (a) Suppose the company decides to increase production by 4 units per day when the current daily production level is 50 units. Find the rate of change of cost with respect to time.
   
   (b) Find each interval on which average cost is increasing and decreasing. For what production level \( x \) is average cost minimized?

12. The demand function for a certain product is given by
\[ p(x) = -0.02x + 400, \quad 0 \leq x \leq 20,000, \] where \( p \) is the unit price when \( x \) items are sold. The cost function for the product is
\[ C(x) = 100x + 300,000. \]
   
   (a) Find the marginal profit of the product when \( x = 2000 \).
   
   (b) Find the actual profit from the sale of the 2001st item. Compare to your answer in (b).
   
   (c) Find each interval on which the profit function is increasing and decreasing. Remember that \( 0 \leq x \leq 20,000 \). How many items should be sold to maximize profit? At what price?

13. Find all relative extrema of \( f(x) = 2x^{5/3} - 5x^{2/3} \). Then find the absolute extrema of \( f(x) \) on \([-8, 0]\). Compare the two methods.
14. Find the absolute maximum and minimum values of \( f(x) = e^{x^3 - 12x} \) on \([0, 3]\).

15. Find the maximum and minimum values of \( f(x) = x^2 - 8 \ln x \) on \([1, e]\).

16. The position (in centimeters) of a particle moving in a straight line at time \( t \) (in seconds) is given by \( s(t) = t^3 - 6t^2 + 9t \) for \( 0 \leq t \leq 6 \).

   (a) Find the velocity function \( v(t) \).

   (b) At what time(s) is the particle at rest?

   (c) For what time interval(s) over the first six seconds is the particle traveling in a positive direction?

   (d) Find the average velocity from \( t = 0 \) to \( t = 4 \) seconds.

   (e) What is the acceleration of the particle after \( 3/2 \) second? Include units in your answer.

   (f) Find each interval on which the particle is (1) speeding up and (2) slowing down

17. Find each interval on which \( f(x) = e^{1-x^2} \) is concave up and down, and find each inflection point of the graph of \( f \).

18. Find all intervals on which the graph of \( f(x) = \frac{x^4}{4} + 2x^3 + \frac{9}{2}x^2 + 8 \) is both decreasing and concave up.

19. Find each interval on which \( f(x) = \ln x + \frac{1}{x} \) is both increasing and concave down. Find each inflection point.

20. Suppose that \( f(x) \) has horizontal tangent lines at \( x = -2, x = 1 \) and \( x = 5 \). If \( f''(x) > 0 \) on intervals \(( -\infty, 0)\) and \(( 2, \infty)\) and \( f''(x) < 0 \) on the interval \(( 0, 2)\), find the \( x \)-values at which \( f(x) \) has relative extrema. Assume that \( f \) and \( f' \) are continuous on \(( -\infty, \infty)\) and use the Second Derivative test.
21. Given the graph of the derivative $f'(x)$, find each interval on which the function $f(x)$ is increasing and decreasing, and find the $x$-coordinate of each point at which $f(x)$ has a local maximum or minimum value. Find each interval on which $f(x)$ is concave up and down, and the $x$-coordinate of each inflection point. Assume that the domain of $f(x)$ is $(-\infty, \infty)$.

![Graph of $y = f'(x)$]

22. Suppose that $f'(x) = 15x^4 - 15x^2$. Find each $x$-value at which the function $f(x)$ has relative extrema. Find the $x$-coordinate of each inflection point. Sketch a possible graph of $f(x)$ if $f(x)$ passes through the origin.

23. A drug that stimulates reproduction is introduced into a population of viruses. That population can be modeled by the function $P(t) = 30t^2 - t^3 + 200$, $0 \leq t \leq 30$, where $P(t)$ is the population after $t$ minutes.

   (a) At what time does the population reach its maximum? What is the maximum population?

   (b) At what time is the rate of growth of the population maximized?
24. A farmer wishes to fence an area next to his barn. He needs a wire fence that costs $1 per linear foot in front of the barn and wooden fencing that costs $2 per foot on the other sides. Find the lengths $x$ (sides perpendicular to the barn) and $y$ (side across from the barn) so that he can enclose the maximum area if his budget for materials is $4400.

25. Frye’s Electronics has started selling a new video game. In one of its Dallas stores, an average of 50 games sell per month at the regular price of $40. The manager of the department has observed that when the video game is put on sale, an average of 5 more games will sell for each $2 price decrease. If each video game costs the store $24 and there are fixed costs of $5600, how many should be sold in a given month to maximize profit? What price should they charge? Assume that the demand function $p(x)$ is linear.

26. The revenue $R(x)$ generated from sales of a certain product is related to the amount of money spent on advertising according to the model $R(x) = \frac{1}{10,000}(600x^2 - x^3)$, $0 \leq x \leq 600$, where $x$ and $R(x)$ are measured in thousands of dollars. Find each interval over which $R(x)$ is increasing. For the interval on which $R(x)$ is increasing, find the point of diminishing returns. Why is it significant?

27. Consider the function $f(x) = x^{1/3}(x + 3)$ and its first two derivatives, $f'(x) = \frac{4x + 3}{3x^{2/3}}$ and $f''(x) = \frac{4x - 6}{9x^{5/3}}$. Find all intercepts, asymptotes, relative extrema and inflection points. Sketch the graph of $f(x)$.

28. Sketch the graph of $f(x) = \frac{x^3}{1 - x^2}$ if $f'(x) = \frac{x^2(3 - x^2)}{(1 - x^2)^2}$ and $f''(x) = \frac{2x(x^2 + 3)}{(1 - x^2)^3}$. 
29. Given the graph of the derivative $f'(x)$, find a possible graph of the function $f(x)$. Assume that $f(-1) = -2$, $f(0) = 0$, $f(1) = 1$, $f(2) = 3$ and $f(3) = 5$, and that $f(x)$ is a continuous function. Be sure to find all extrema and inflection points.
1. The number of gallons of water in a tank \( t \) minutes after the tank has started to drain is given by \( Q(t) = 200(30 - t)^2 \).

(a) How much water is in the tank 10 minutes after it has started to drain?
(b) How fast is the water running out of the tank at \( t = 10 \) minutes?
(c) Use parts (a) and (b) to approximate the amount of water in the tank one minute later \((t = 11)\). Compare to the actual amount of water in the tank at that time.

2. A pollutant from a factory is carried away by wind currents. Its concentration in the air \( x \) miles from the factory is given by \( P(x) = \frac{1.6}{3x - 2} \) where \( P(x) \) is measured in parts per million.

(a) What is the average rate at which the concentration is changing when the pollutant has moved from 1 to 4 miles away from the factory? Include units in your answer.
(b) At what rate is the concentration changing when the pollutant is two miles from the factory? Include units.
(c) Now suppose that the distance of the pollutant from the factory is given by \( x(t) = t^2 + t \) where \( t \) is in hours. Find the rate at which the concentration of the pollutant is changing with respect to time after 2 hours.

3. The cost function for a product is \( C(x) = 1.25x^2 + 25x + 8000 \).

(a) Suppose the company decides to increase production by 4 units per day when the current daily production level is 50 units. Find the rate of change of cost with respect to time.
(b) Average cost is defined to be \( \frac{C(x)}{x} \). Find and interpret marginal average cost at a production level of 50 units.

4. The demand function for a product is \( p(x) = 45 - \sqrt{\frac{x}{2}} \) where \( p \) is the price at which \( x \) items will sell.

(a) Find the marginal revenue when 1600 items are produced.
(b) Now suppose that when 1600 items are produced in a week, revenue increases by $450 per week. At what rate is the production level \( x \) changing with respect to time at that production level?

5. Sand is falling off of a conveyor belt in a conical shape at the rate of 30 ft\(^3\)/min. in such a way that the radius of the cone remains approximately three-fourths of its height. At what rate is the height of the pile changing when it is 8 feet high? Recall that the volume of a cone is given by \( V(r) = \frac{1}{3}\pi r^2 h \).

6. An observer is standing 50 feet from a helicopter launch pad. A helicopter lifts off vertically and rises at a rate of 40 ft/sec. At what rate is the distance between the observer and the helicopter changing at the instant when the helicopter is 120 feet high?